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# Interval Linear Programming with generalized interval arithmetic

G. Ramesh, K. Ganesan

**Abstract**— Generally, vagueness is modelled by a fuzzy approach and randomness by a stochastic approach. But in some cases, a decision maker may prefer using interval numbers as coefficients of an inexact relationship. In this paper, we define a linear programming problem involving interval numbers as an extension of the classical linear programming problem to an inexact environment. By using a new simple ranking for interval numbers and new generalized interval arithmetic, we attempt to develop a theory for solving interval number linear programming problems without converting them to classical linear programming problems.

**Index Terms**— Interval Numbers, generalized interval arithmetic, Interval Linear Programming, Ranking

## 1 INTRODUCTION

IN order to solve a Linear Programming Problem, the decision parameters of the model must be fixed at crisp values. But in real-world applications certainty, reliability and precision of data is often not possible and it involves high information costs. Furthermore the optimal solution of a linear programming problem depends on a limited number of constraints and thus some of the information collected are not useful. Hence in order to reduce information costs and also to construct a real model, the use of interval linear programs are more appropriate.

Linear programming problems with interval coefficients have been studied by several authors, such as Atanu Sengupta et al. [3] , Bitran [4], Chanas and Kuchta [5], Ishibuchi and Tanaka [12, 13], Lodwick and Jamison [15], Nakahara et al. [16], Steuer [23], Chinneck and Ramadan [6], Herry Suprajitno and Ismail bin Mohd [10] and Tong Shaocheng [24]. Numerous methods for comparison of interval numbers can be found as in Atanu Sengupta [2] etc.

Chanas and Kuchta [5] generalized the known concepts of the solution of linear programming problem with interval coefficients in the objective function based on preference relaions between intervals. Tong Shaocheng [19] reduced the interval number linear programming problem into two classical linear programming problems by taking maximum value range and minimum value range inequalities as constraint conditions and then obtained an optimal interval solution.

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Sengupta et al [3] have reduced the interval number linear programming problem into a biobjective classical linear programming problem and then obtained an optimal solution. Ishibuchi and Tanaka [12, 13] have defined two types of partial order relations $\leq_{LR}$ and $\leq_{mw}$ between two interval numbers. Based on these partial order relations, they have studied the linear programming problems with interval coefficients in the objective function, by transforming into a standard bi-objective optimization problem.

Oliveira and Antunes [22] provide an illustrated overview of the state of the art of interval programming in the context of multiple objective linear programming models. Mraz [19] computes the exact upper and lower bounds of optimal values for linear programming problems whose coefficients vary in a given intervals. Hladik [11] computes exact range of the optimal value for linear programming problem in which input data can vary in some given real compact intervals, and he able to characterize the primal and dual solution sets, the bounds of the objective function resulted from two nonlinear programming problems. Herry Suprajitno and Ismail bin Mohdwe [10] proposed a modification of simplex method for the solution of interval linear programming problems using the software Pascal-XSC.

In general, they have transformed the interval linear programming problems into one or a series of classical linear programming problems and then obtained an optimal solution.

But in this paper, by using a new method of comparison of generalized interval numbers and a new set of generalized interval arithmetic [7, 8, 17], we develop a simplex like algorithm for solving interval linear programming problems without converting them to classical linear programming problems.

The rest of this paper is organized as follows: In section 2, we recall the definition of a new type of arithmetic operations, a linear order relation between interval numbers and some related results. In section 3, we define interval number linear programming problem as an extension of the classical linear programming problem to an inexact environment and prove some basic results of linear programming problems involving generalized interval numbers.

## 2 Preliminaries

The aim of this section is to present some notations, notions and results which are of useful in our further consideration.

Let $\tilde{a}=[a_1, a_2]=\{x: a_1 \le x \le a_2, x \in R\}$. If $\tilde{a}=a_1=a_2=a$, then $\tilde{a}=[a, a]=a$ is a real number (or a degenerate interval). Let $IR=\{\tilde{a}=[a_1, a_2]: a_1 \le a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all proper intervals and $\overline{IR}=\{\tilde{a}=[a_1, a_2]: a_1 > a_2 \text{ and } a_1, a_2 \in R\}$ be the set of all improper intervals on the real line R. We shall use the terms"interval" and"interval number" interchangeably.

The mid-point and width (or half-width) of an interval number $\tilde{a}=[a_1, a_2]$ are defined as

$$m(\tilde{a})=\left(\frac{a_1+a_2}{2}\right) \quad \text{and .} \quad w(\tilde{a})=\left(\frac{a_2-a_1}{2}\right)$$

Algebraic properties of classical interval arithmetic defined on IR are often insufficient if we want to deal with real world problems involving interval parameters. Because, intervals with nonzero width do not have inverses in IR with respect to the classical interval arithmetical operations. This "incompleteness" stimulated attempts to create a more convenient interval arithmetic extending that based on IR. In this direction, several extensions of the classical interval arithmetic have been proposed. Kaucher [13] proposed a new method, in which the set of proper intervals is extended by improper intervals and the interval arithmetic operations and functions are extended correspondingly. We denote the set of generalized intervals (proper and improper) by

$$KR=IR \cup \overline{IR}=\{[a_1, a_2]: a_1, a_2 \in R\}.$$

The set of generalized intervals KR is a group with respect to addition and multiplication operations of zero free intervals, while maintaining the inclusion monotonicity. The algebraic properties of the generalized interval arithmetic create a suitable environment for solving algebraic problems involving interval numbers. However, the efficient solution of some interval algebraic problems is hampered by the lack of well studied distributive relations between generalized intervals. Ganesan and Veeramani [7] proposed a new interval arithmetic which satisfies the distributive relations between intervals.

The"dual" is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in KR. For $\tilde{a}=[a_1, a_2] \in KR$, its dual is defined by $dual(\tilde{a})=dual([a_1, a_2])=[a_2, a_1]$. The opposite of an interval $\tilde{a}=[a_1, a_2]$ is $opp(\tilde{a})=opp([a_1, a_2])=[-a_1, -a_2]$ which is the additive inverse of $\tilde{a}=[a_1, a_2]$ and $\frac{1}{\tilde{a}}=\left[\frac{1}{a_1}, \frac{1}{a_2}\right]$ is the multiplicative inverse of $\tilde{a}=[a_1, a_2]$ provided $0 \notin [a_1, a_2]$

### 2.1 Ranking of Interval Numbers

Sengupta and Pal [2] proposed a simple and efficient index for comparing any two intervals on IR through decision maker's satisfaction. We extend this concept to the set of all generalized intervals on KR.

**Definition 2.1.** Let $\prec$ be an extended order relation between the interval numbers $\tilde{a}=[a_1, a_2]$ and $\tilde{b}=[b_1, b_2]$ in IR, then for $m(\tilde{a})<m(\tilde{b})$, we construct a premise $(\tilde{a} \prec \tilde{b})$ which implies that $\tilde{a}$ is inferior to $\tilde{b}$ (or $\tilde{b}$ is superior to $\tilde{a}$).

An *acceptability function* $A_{\preceq}: KR \times KR \to [0, \infty)$ is defined as:

$$A_{\preceq}(\tilde{a}, \tilde{b})=A(\tilde{a} \preceq \tilde{b})=\frac{m(\tilde{b})-m(\tilde{a})}{w(\tilde{b})+w(\tilde{a})}, \quad \text{where } w(\tilde{b})+w(\tilde{a}) \neq 0.$$

$A_{\preceq}$ may be interpreted as the grade of acceptability of the "first interval number to be inferior to the second interval number".

For any two interval numbers $\tilde{a}$ and $\tilde{b}$ in KR, either

$$A(\tilde{a} \preceq \tilde{b}) \ge 0 \text{ or } A(\tilde{b} \preceq \tilde{a}) \ge 0 \text{ or } A(\tilde{a} \preceq \tilde{b})=A(\tilde{b} \preceq \tilde{a})=0$$
$$\text{and } A(\tilde{a} \preceq \tilde{b})+A(\tilde{b} \preceq \tilde{a})=0.$$

Also the prposed $A$ - index is transitive; for any three interval numbers $\tilde{a}, \tilde{b}, \tilde{c}$ in KR, if

$$A(\tilde{a} \preceq \tilde{b}) \ge 0 \text{ and } A(\tilde{b} \preceq \tilde{c}) \ge 0, \text{then } A(\tilde{a} \preceq \tilde{c}).$$

But it does not mean that

$$A(\tilde{a} \preceq \tilde{c}) \ge \max\{A(\tilde{a} \preceq \tilde{b}), A(\tilde{b} \preceq \tilde{c})\}.$$

If $A(\tilde{a} \preceq \tilde{b})=0$ then we say that the interval numbers $\tilde{a}$ and $\tilde{b}$ are equivalent (or non-inferior to each other) and we denote it by $\tilde{a} \approx \tilde{b}$. In particular, whenever $A(\tilde{a} \preceq \tilde{b})=0$ and $w(\tilde{b})=w(\tilde{a})$, then $\tilde{a}=\tilde{b}$. Also if $A(\tilde{a} \preceq \tilde{b}) \ge 0$, then we say that $\tilde{a} \prec \tilde{b}$ and if $A(\tilde{b} \preceq \tilde{a}) \ge 0$ , then we say that $\tilde{b} \preceq \tilde{a}$.

### 2.2 A New Interval Arithmetic

Ganesan and Veeramani [3] proposed new interval arithmetic on IR. We extend this arithmetic operation to the set of generalized interval numbers KR and incorporating the concept of dual. For $\tilde{a}=[a_1, a_2]$, $\tilde{b}=[b_1, b_2] \in KR$ and for $* \in \{+, -, \cdot, \div\}$,

we define $\tilde{a} * \tilde{b}=[m(\tilde{a}) * m(\tilde{b})-k,\ m(\tilde{a}) * m(\tilde{b})+k]$,
where $k=\min\left\{\left((m(\tilde{a}) * m(\tilde{b}))-\alpha\right), \beta-\left((m(\tilde{a}) * m(\tilde{b}))\right)\right\}$,

$\alpha$ and $\beta$ are the end points of the interval $\tilde{a} * \tilde{b}$ under the existing interval arithmetic. In particular

**(i). Addition :** $\tilde{a}+\tilde{b}=[a_1, a_2]+[b_1, b_2]$

$$=[\{m(\tilde{a})+m(\tilde{b})\}-k,\ \{m(\tilde{a})+m(\tilde{b})\}+k]$$

$$\text{where } \quad k=\min\left(\frac{(b_2+a_2)-(b_1+a_1)}{2}\right).$$

**(ii). Subtraction :** $\tilde{a} - \tilde{b} = [a_1, a_2] - [b_1, b_2]$

$$= [\{m(\tilde{a}) - m(\tilde{b})\} - k,\ \{m(\tilde{a}) - m(\tilde{b})\} + k]$$

where $k = \min\left(\frac{(b_2 + a_2) - (b_1 + a_1)}{2}\right)$.

Also if $\tilde{a} = \tilde{b}$ i.e. $[a_1, a_2] = [b_1, b_2]$, then

$$\tilde{a} - \tilde{b} = \tilde{a} - \text{dual}(\tilde{a}) = [a_1, a_2] - \text{dual}([a_1, a_2])$$
$$= [a_1, a_2] - [a_2, a_1] = [a_1 - a_1, a_2 - a_2] = [0, 0]$$

**(iii). Multiplication :** $\tilde{a}\tilde{b} = [a_1, a_2][b_1, b_2]$

$$= \left[\left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) - w, \left(\frac{a_1 + a_2}{2}\right)\left(\frac{b_1 + b_2}{2}\right) + w\right]$$

where $w = \left(\frac{\beta - \alpha}{2}\right)$, $\alpha = \min(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$

and $\beta = \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)$

Also if $\tilde{a} = \tilde{b}$ i.e. $[a_1, a_2] = [b_1, b_2]$, then

$$\frac{\tilde{a}}{\tilde{b}} = \frac{\tilde{a}}{\text{dual}(\tilde{a})} = \tilde{a} \times \left(\frac{1}{\text{dual}\ \tilde{a}}\right) = [a_1, a_2] \times \frac{1}{\text{dual}([a_1, a_2])}$$
$$= [a_1, a_2] \times \frac{1}{[a_2, a_1]} = [a_1, a_2] \times \left[\frac{1}{a_1}\ \frac{1}{a_2}\right]$$
$$= \left[\frac{a_1}{a_1}\ \frac{a_2}{a_2}\right] = [1, 1]$$

**(iv). Division :** $\tilde{a}^{-1} = [a_1, a_2]^{-1} = \left[\frac{1}{m(\tilde{a})} - k,\ \frac{1}{m(\tilde{a})} + k\right]$,

where $k = \min\left\{\frac{1}{a_2}\left(\frac{a_2 - a_1}{a_1 + a_2}\right), \frac{1}{a_1}\left(\frac{a_2 - a_1}{a_1 + a_2}\right)\right\}$

From (iii), it is clear that $\lambda\tilde{a} = \begin{cases}[\lambda a_1, \lambda a_2], & \text{for } \lambda \geq 0\\ [\lambda a_2, \lambda a_1], & \text{for } \lambda < 0\end{cases}$

An interval number $\tilde{a}$ is said to be positive if and only if $\tilde{a} \succ \tilde{0}$. That is $\tilde{a} \succeq \tilde{0}$ and $\tilde{a} \not\approx [-\alpha, \alpha]$ for each $\alpha \geq 0$. Also if $\tilde{a} \approx \tilde{0}$ then $\tilde{a}$ is said to be a zero interval number. It is to be noted that if $\tilde{a} = \tilde{0}$, then $\tilde{a} \approx \tilde{0}$, but the converse need not be true. If $\tilde{a} \not\approx \tilde{0}$, then $\tilde{a}$ is said to be a non-zero interval number. It is to be noted that if $\tilde{a} \not\approx \tilde{0}$, then $\tilde{a} \neq \tilde{0}$, but the converse need not be true.

Now we recall some of the results from [7], which will be useful in proving important theorems.

**Proposition 2.1.** If $\tilde{a}$, $\tilde{b}$ are two interval numbers with $(\tilde{a} - \tilde{b}) \approx \tilde{0}$, then for any interval number $\tilde{c}$, we have $\tilde{c}\tilde{a} \approx \tilde{c}\tilde{b}$.

**Proposition 2.2**. If $\tilde{a}$, $\tilde{b}$ are two interval numbers with $(\tilde{a} - \tilde{b}) \preceq \tilde{0}$, then for any interval number $\tilde{c} \succeq \tilde{0}$, we have $\tilde{c}\tilde{a} \preceq \tilde{c}\tilde{b}$.

**Proposition 2.3.** If $\tilde{a}$, $\tilde{b}$ are two interval numbers with $(\tilde{a} - \tilde{b}) \preceq \tilde{0}$, then for any interval number $\tilde{c} \preceq \tilde{0}$, we have $\tilde{c}\tilde{a} \succeq \tilde{c}\tilde{b}$.

**Proposition 2.4**. For any three interval numbers $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2]$ and $\tilde{c} = [c_1, c_2]$, we have

(i) $\tilde{c}(\tilde{a} + \tilde{b}) \approx (\tilde{c}\tilde{a} + \tilde{c}\tilde{b})$ and

(ii) $\tilde{c}(\tilde{a} - \tilde{b}) \approx (\tilde{c}\tilde{a} - \tilde{c}\tilde{b})$.

It is to be noted that, if we use the existing multiplication rule for interval numbers,

$$\tilde{a}\tilde{b} = [a_1, a_2][b_1, b_2] = \left[\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2),\ \max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)\right]$$

the above proposition need not be true. For example, let $\tilde{a} = [-1, 2]$, $\tilde{b} = [2, 5]$ and $\tilde{c} = [2, 3]$ are any three interval numbers. Then by using the existing interval arithmetic, we have $(\tilde{a} + \tilde{b}) = [1, 7]$ and $\tilde{c}(\tilde{a} + \tilde{b}) = [2, 21]$. Also $\tilde{c}\tilde{a} = [-3, 6]$ and $\tilde{c}\tilde{b} = [4, 15] \Rightarrow (\tilde{c}\tilde{a} + \tilde{c}\tilde{b}) = [1, 21]$. So that $\tilde{c}(\tilde{a} + \tilde{b}) \not\approx (\tilde{c}\tilde{a} + \tilde{c}\tilde{b})$.

## 3 Main Results

In this paper we consider linear programming problems involving interval numbers as follows:

Let KR be the set of all generalized interval numbers. Consider the following linear programming problem involving interval numbers

$$\max \tilde{z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j$$

subject to $\sum_{j=1}^{n} a_{ij}\tilde{x}_j \preceq \tilde{b}_i$, $i = 1, 2, 3 \ldots m$ , (3.1)

and $\tilde{x}_j \geq \tilde{0}$ for all j=1,2,3,……..,n, where $a_{ij} \in R$, $\tilde{c}_j, \tilde{x}_j, \tilde{b}_i \in KR$, , $i = 1, 2, 3, \ldots\ldots m$ and $j = 1, 2, 3, \ldots\ldots, n$.

We call the above problem (3.1) as an interval number linear programming problem and it can be rewritten as

$$\max \tilde{z} \approx \tilde{\mathbf{c}}\tilde{\mathbf{x}} \quad \text{subject to } A\,\tilde{\mathbf{x}} \preceq \tilde{\mathbf{b}} \text{ and } \tilde{\mathbf{x}} \succeq \tilde{\mathbf{0}} \qquad (3.2)$$

where A is an $(m \times n)$ real matrix and $\tilde{\mathbf{b}}, \tilde{\mathbf{c}}, \tilde{\mathbf{x}}$ are $(m \times 1)$, $(1 \times n)$, $(n \times 1)$ matrices consisting of interval numbers.

Let $X = \{\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \ldots\ldots, \tilde{x}_n) : A\tilde{\mathbf{x}} \preceq \tilde{\mathbf{b}},\ \tilde{\mathbf{x}} \succeq \tilde{\mathbf{0}}\}$ be the feasible region of problem (3.1). A feasible solution $\tilde{\mathbf{x}}^* \in \mathbf{X}$, is said to be an optimum solution to (3.1) if $\tilde{\mathbf{c}}\tilde{\mathbf{x}}^* \succeq \tilde{\mathbf{c}}\tilde{\mathbf{x}}$ for all $\tilde{\mathbf{x}} \in \mathbf{X}$, where $\tilde{\mathbf{c}} = (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3, \ldots\ldots, \tilde{c}_n)$ and $\tilde{\mathbf{c}}\tilde{\mathbf{x}} = \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j$.

**Standard Form**:
For the general study, we convert the given interval number linear programming problem into its standard form:

$$\max \tilde{z} \approx \tilde{c}\tilde{x} \quad \text{subject to } A\tilde{x} \approx \tilde{b} \text{ and } \tilde{x} \succeq \tilde{0}, \qquad (3.3)$$

where A is an (m× n) real matrix, $\tilde{b}, \tilde{c}, \tilde{x}$ are (m××1), (1××n), (n××1) matrices consisting of interval numbers.

**Definition 3.1.** Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_n)$. Suppose $\tilde{x}$ solves $A\tilde{x} \approx \tilde{b}$. If all $\tilde{x}_j \approx [-\alpha_j, \alpha_j]$ for some $\alpha_j \geq 0$, then $\tilde{x}$ is said to be a basic solution. If $\tilde{x}_j \not\approx [-\alpha_j, \alpha_j]$ for some $\alpha_j \geq 0$, then $\tilde{x}$ has some non-zero components, say $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ......, \tilde{x}_k$, $1 \leq k \leq n$. Then $A\tilde{x} \approx \tilde{b}$ can be written as:

$$\mathbf{a}_1\tilde{x}_1 + \mathbf{a}_2\tilde{x}_2 + \mathbf{a}_3\tilde{x}_3 + ... + \mathbf{a}_k\tilde{x}_k + \mathbf{a}_{k+1}[-\beta_{k+1}, \beta_{k+1}] + \mathbf{a}_{k+2}[-\beta_{k+2}, \beta_{k+2}] + ... + \mathbf{a}_n[-\beta_n, \beta_n] \approx \tilde{b}$$

If the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, ..., \mathbf{a}_k$ corresponding to these non-zero components $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ......, \tilde{x}_k$ are linearly independent, then $\tilde{x}$ is said to be a basic solution.

**Remark 3.1**. Given a system of *m* simultaneous linear equations involving interval numbers in n unknowns (m ≤ n) $A\tilde{x} \approx \tilde{b}$, $\tilde{b} \in KR^m$, where A is a (m× n) real matrix and rank of A is m. Let B be any (m× m) matrix formed by m linearly independent columns of A.

Let $\tilde{x}_B = B^{-1}\tilde{b} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 ..., \tilde{x}_m)^T$ simply we write $\tilde{x}_B = B^{-1}\tilde{b} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 ..., \tilde{x}_m)$ then $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_m, \tilde{0}, \tilde{0}, ..., \tilde{0})$ is a basic solution. In this case, we also say that $\tilde{x}_B$ is a basic solution.

**Remark 3.2.** Consider $A\tilde{x} \approx \tilde{b}$ where $A = (a_{ij})_{m \times n}$, $a_{ij} \in R$. Then $\tilde{x}_B = B^{-1}\tilde{b}$ is a Solution of $A\tilde{x} \approx \tilde{b}$.

Now we are in a position to prove some important theorems on interval number linear Programming problems.

**Theorem 3.1.** If there is any feasible solution to the problem (3.3), then there is a basic feasible solution to (3.3).

**Proof.** Let $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ......., \tilde{x}_n)$ be a feasible solution to the problem (3.3) with the fewest positive components such that

$$A\tilde{x} \approx \mathbf{a}_1\tilde{x}_1 + \mathbf{a}_2\tilde{x}_2 + \mathbf{a}_3\tilde{x}_3 + ... + \mathbf{a}_n\tilde{x}_n \approx \tilde{b} \qquad (3.4)$$

**Case (i):** If $\tilde{x}$ has no positive components, then $\tilde{x}_j \approx [-\alpha_j, \alpha_j]$, $\alpha_j \in R$, $\alpha_j \geq 0$ for each j and $\tilde{x}$ is basic by definition.

**Case(ii):** If $\tilde{x}$ has k positive components, say $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ......, \tilde{x}_k$, $1 \leq k \leq n$, then $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_k, \tilde{x}_{k+1}, ..., \tilde{x}_n)$, where $\tilde{x}_j = [\alpha_j, \beta_j]$, $\alpha_j \leq \beta_j$, for j=1, 2, 3,…,n and

$$\frac{\alpha_j + \beta_j}{2} > 0, \text{ for j=1, 2, 3,…, k;}$$

$$\frac{\alpha_j + \beta_j}{2} = 0, \text{ for j=k+1, k+2,…,n.}$$

That is $\tilde{x}_j \succ 0$, for j = 1, 2, 3, $\cdots$, k;

$\tilde{x}_j = [-\beta_j, \beta_j]$, for $j = k+1, k+2, n$, and

$\beta_j \geq 0$, for all j = 1, 2, 3, $\cdots$, n.

Now equation (3.4) becomes

$$\sum_{j=1}^{k} \tilde{x}_j \mathbf{a}_j + [-\beta_{k+1}, \beta_{k+1}]\mathbf{a}_{k+1} + [-\beta_{k+2}, \beta_{k+2}]\mathbf{a}_{k+2} + ... + [-\beta_n, \beta_n]\mathbf{a}_n \approx \tilde{b}$$

That is

$$\sum_{j=1}^{k} \tilde{x}_j \mathbf{a}_j + \sum_{j=k+1}^{n} [-\beta_j, \beta_j]\mathbf{a}_j \approx \tilde{b}. \qquad (3.5)$$

Suppose that the columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, ..., \mathbf{a}_k$ corresponding to these positive components $\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, ..., \tilde{x}_k$ are linearly dependent, then for atleast one $\theta_j \neq 0$ (say $\theta_r \neq 0$), we have

$$\sum_{j=1}^{k} \theta_j a_j = 0 \text{ and hence } a_r = -\sum_{j=1}^{k} \left(\frac{\theta_j}{\theta_r}\right) a_j, \text{ for } j \neq r.$$

Then equation (3.5) becomes

$$\sum_{\substack{j=1 \\ j \neq r}}^{k} \tilde{x}_j \mathbf{a}_j - \sum_{\substack{j=1 \\ j \neq r}}^{k} \left(\frac{\theta_j}{\theta_r}\right) \tilde{x}_r \mathbf{a}_j + \sum_{j=k+1}^{n} [-\beta_j, \beta_j] \mathbf{a}_j \approx \tilde{b}$$

$$\Rightarrow \sum_{\substack{j=1 \\ j \neq r}}^{k} \left(\tilde{x}_j - \frac{\theta_j}{\theta_r}\tilde{x}_r\right) \mathbf{a}_j + \sum_{j=k+1}^{n} [-\beta_j, \beta_j] \mathbf{a}_j \approx \tilde{b}$$

$$\sum_{\substack{j=1 \\ j \neq r}}^{k} \left(\tilde{x}_j - \tilde{x}_r\right)\frac{\theta_j}{\theta_r} \mathbf{a}_j + \left(\tilde{x}_j - \frac{\theta_j}{\theta_r}\tilde{x}_r\right) \mathbf{a}_r + \sum_{j=k+1}^{n} [\beta, \beta_j] \mathbf{a}_j \approx \tilde{b}$$

That is

$$\sum_{\substack{j=1 \\ j \neq r}}^{k} \left(\tilde{x}_j - \frac{\theta_j}{\theta_r}\tilde{x}_r\right) \mathbf{a}_j + [-\beta_r, \beta_r] \mathbf{a}_r + \sum_{j=k+1}^{n} [-\beta_j, \beta_j] \mathbf{a}_j \approx \tilde{b}. \qquad (3.6)$$

We are not sure that these variables $\left(\tilde{x}_j - \frac{\theta_j}{\theta_r}\tilde{x}_r\right)$, for $j \neq r$ are non-negative. To ensure that these are non-negative, we must have $\left(\tilde{x}_j - \frac{\theta_j}{\theta_r}\tilde{x}_r\right) \succeq \tilde{0}$, for j=r

Now select $\theta_r > 0$, such that $\frac{\tilde{x}_r}{\theta_r} \approx min\left\{\frac{\tilde{x}_j}{\theta_j} : \tilde{x}_j > \tilde{0}, \theta_j > 0\right\}$.

Then $\frac{\tilde{x}_r}{\theta_r} \preceq \frac{\tilde{x}_j}{\theta_j} \Rightarrow \left[\frac{\alpha_r}{\theta_r}, \frac{\beta_r}{\theta_r}\right] \preceq \left[\frac{\alpha_j}{\theta_j}, \frac{\beta_j}{\theta_j}\right]$

$$\Rightarrow \left[\frac{\alpha_j}{\theta_j}, \frac{\beta_j}{\theta_j}\right] - \left[\frac{\alpha_r}{\theta_r}, \frac{\beta_r}{\theta_r}\right] \succeq \tilde{0}, \text{ by proposition (2.4).}$$

$$\Rightarrow \left[\frac{\alpha_j}{\theta_j} - \frac{\beta_r}{\theta_r}, \frac{\beta_j}{\theta_j} - \frac{\alpha_r}{\theta_r}\right] \succeq [-\alpha, \alpha]$$

$$\Rightarrow \left\{\frac{\left(\frac{\alpha_j}{\theta_j} - \frac{\beta_r}{\theta_r}\right) + \left(\frac{\beta_j}{\theta_j} - \frac{\alpha_r}{\theta_r}\right)}{2}\right\} \geq 0$$

$$\Rightarrow\left(\frac{\alpha_i+\beta_j}{\theta_j}\right)-\left(\frac{\alpha_r+\beta_r}{\theta_r}\right)\geq 0 \Rightarrow \left(\frac{\alpha_i+\beta_j}{\theta_j}\right)\geq\left(\frac{\alpha_r+\beta_r}{\theta_r}\right)$$

$$\Rightarrow\left(\frac{\tilde{x}_j}{\theta_j}-\frac{\tilde{x}_r}{\theta_r}\right)\succeq\tilde{0}\Rightarrow\left(\tilde{x}_j-\frac{\theta_j}{\theta_r}\tilde{x}_r\right)\succeq\tilde{0},\ \text{for}\ \ j\neq r\ .$$

Hence each $\left(\tilde{x}_j-\dfrac{\theta_j}{\theta_r}\tilde{x}_r\right)$ is positive for $j \neq r$ and we have a new feasible solution with (k - 1) positive variables. That is we have a new feasible solution with fewer positive components than $\tilde{\mathbf{x}}$ has. The new rth component is of the form $[-\beta_r, \beta_r]$, $\beta_r \in R$, whereas the old rth component $\tilde{x}_r \succ \tilde{0}$. This contradicts our assumption that $\tilde{\mathbf{x}}$ was a feasible solution with fewest possible positive components. Therefore the columns $\mathbf{a_1, a_2, a_3, ..., a_k}$ are linearly independent. Hence the feasible solution $\tilde{\mathbf{x}}$ is basic.

**Remark 3.3.** It is to be noted that for an interval number linear programming problem, there are infinite number of basic solutions. But the number of non-equivalent basic solutions is finite.

**Proposition 3.1**. Let $\tilde{\mathbf{x}}$ be a basic feasible solution to problem (3.3) and $\tilde{\mathbf{y}}$ is such that $(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\approx\tilde{\mathbf{0}}$, then $\tilde{\mathbf{y}}$ is also a basic feasible solution to (3.3).

**Theorem 3.2**. If there is any optimal solution to problem (3.3), then there is a basic optimal solution to (3.3).

**Proof.** Let $\tilde{\boldsymbol{x}}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,......,\tilde{x}_n)$ be an optimal solution to problem (3.3) with fewest possible positive components and $\tilde{\boldsymbol{z}}_0\approx\tilde{\boldsymbol{c}}\tilde{\boldsymbol{x}}$ be the corresponding optimal value.

**Case(i)** If $\tilde{\mathbf{x}}$ has no positive components, then each $\tilde{x}_j\approx[-\alpha_j,\alpha_j]$, $\alpha_j\in R, \alpha_j\geq 0$ and $\tilde{\mathbf{x}}$ is basic by definition.

**Case(ii)** If $\tilde{\mathbf{x}}$ has k positive components say $\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,......,\tilde{x}_k$, $1\leq k\leq n$, then

$\tilde{\mathbf{x}}=(\tilde{x}_1,\tilde{x}_2,\tilde{x}_3,......,\tilde{x}_k,[-\beta_{k+1}, \beta_{k+1}],[-\beta_{k+2},\beta_{k+2}],...,[-\beta_n,\beta_n])$ so that $\sum\limits_{j=1}^{k}\tilde{x}_j\,\mathbf{a_j}+\sum\limits_{j=k+1}^{n}[-\beta_j,\beta_j]\,\mathbf{a_j}\approx\tilde{\mathbf{b}}$

The corresponding optimal cost is given by

$$\tilde{z}_0\approx\sum_{j=1}^{k}\tilde{c}_j\tilde{x}_j+\sum_{j=k+1}^{n}\tilde{c}_j[-\beta_j,\beta_j]. \qquad (3.7)$$

Since $\tilde{\mathbf{x}}$ is an optimal solution with k positive components, it is a feasible solution with k positive components. If $\tilde{\mathbf{x}}$ is not a basic feasible solution, then by theorem (3.1), we have a new feasible solution with $(k-1)$ positive components $\left(\tilde{x}_j-\dfrac{\theta_j}{\theta_r}\tilde{x}_r\right)$, for *j = 1, 2, 3,…, k* and *j≠ r.* The cost corresponding to this new feasible solution is

$$\tilde{z}^*\approx\sum_{\substack{j=1\\j\neq r}}^{k}\tilde{c}_j\left(\tilde{x}_j-\frac{\theta_j}{\theta_r}\tilde{x}_r\right)+\sum_{j=k+1}^{n}\tilde{c}_j[-\beta_j,\beta_j]$$

$$\approx\sum_{\substack{j=1\\j\neq r}}^{k}\tilde{c}_j\left(\tilde{x}_j-\frac{\theta_j}{\theta_r}\tilde{x}_r\right)+\tilde{c}_r\left(\tilde{x}_r-\frac{\theta_r}{\theta_r}\tilde{x}_r\right)+\sum_{j=k+1}^{n}\tilde{c}_j[-\beta_j,\beta_j]$$

$$\approx\sum_{j=1}^{k}\tilde{c}_j\left(\tilde{x}_j-\frac{\theta_j}{\theta_r}\tilde{x}_r\right)+\sum_{j=k+1}^{n}\tilde{c}_j[-\beta_j,\beta_j]$$

$$\approx\sum_{j=1}^{k}\tilde{c}_j\tilde{x}_j-\frac{\tilde{x}_r}{\theta_r}\sum_{j=1}^{k}\tilde{c}_j\theta_j+\sum_{j=k+1}^{n}\tilde{c}_j[-\beta_j,\beta_j],\ \text{by proposition (2.4)}$$

$$\approx\tilde{z}_0-\frac{\tilde{x}_r}{\theta_r}\sum_{j=1}^{k}\tilde{c}_j\theta_j\ . \qquad (3.8)$$

If $\sum\limits_{j=1}^{k}\tilde{c}_j\theta_j\not\approx[-\alpha_j,\alpha_j]$, $\alpha_j\in R$, then $\tilde{z}^*\succ\tilde{z}_0$ by taking $\left(\dfrac{\tilde{x}_r}{\theta_r}\right)$ to be some positive or negative interval number. This is a contradiction to our assumption that $\tilde{\mathbf{x}}$ is optimal. Hence $\sum\limits_{j=1}^{k}\tilde{c}_j\theta_j\approx[-\alpha_j,\alpha_j]$, $\alpha_j\in R$. Then equation (3.8) becomes $\tilde{z}^*\approx\tilde{z}_0\ \pm[-\delta,\delta]\approx\tilde{z}_0$. The new optimal solution has fewer positive components than the old optimal solution $\tilde{\mathbf{x}}$**.** This contradicts our assumption. Hence the optimal solution $\tilde{\mathbf{x}}$ is basic.

**Proposition 3.2.** Let $\tilde{\mathbf{x}}$ is an optimal solution to problem (3.3) and $\tilde{\mathbf{y}}$ is such that $(\tilde{\mathbf{x}}-\tilde{\mathbf{y}})\approx\tilde{\mathbf{0}}$, then $\tilde{\mathbf{y}}$ is also an optimal solution to (3.3).

### 3.1 Improving a basic feasible solution

Let $B=(\mathbf{b_1, b_2, ..., b_m})$ form a basis for the columns of A. Let $\tilde{\mathbf{x}}_B=B^{-1}\tilde{\mathbf{b}}$ be a basic feasible solution and the value of the objective function $\tilde{z}$ is given by $\tilde{z}_0\approx\tilde{\mathbf{c}}_\mathbf{B}\tilde{\mathbf{x}}_\mathbf{B}$, where $\tilde{c}_B=(\tilde{c}_{B1},\tilde{c}_{B2},...,\tilde{c}_{Bm})$ be the cost vector corresponding to $\tilde{\mathbf{x}}_\mathbf{B}$. Assume that $\mathbf{a_j}=\sum\limits_{i=1}^{m}y_{ij}\mathbf{b_i}=\mathbf{y_j B}$ and the interval number $\tilde{z}_j=\tilde{c}_B y_j$ are known for every column vector $\mathbf{a_j}$ in A, which is not in **B**. Now we shall examine the possibility of finding another basic feasible solution with an improved value of $\tilde{z}$ by replacing one of the columns of **B** by $\mathbf{a_j}$.

**Theorem 3.3.** Let $\tilde{\mathbf{x}}_B=B^{-1}\tilde{\mathbf{b}}$ be a basic feasible solution to problem (3.3). If for any column $\mathbf{a_j}$ in A which is not in **B**, the condition $(\tilde{z}_j-\tilde{c}_j)\prec\tilde{0}$ hold and $y_{ij}>0$ for some *i*, $i\in\{1,2,3,...,m\}$ then it is possible to obtain a new basic feasible solution by replacing one of the columns in **B** by $\mathbf{a_j}$. After the replacement of basis vectors, the new basis matrix is $\hat{B}=(\hat{\mathbf{b}}_1,\hat{\mathbf{b}}_2,...,\hat{\mathbf{b}}_m)$, where $\hat{\mathbf{b}}_i=\mathbf{b}_i$ for $i\neq r$ and $\hat{\mathbf{b}}_r=\mathbf{a_j}$. The new basic feasible solution is $\hat{\tilde{\mathbf{x}}}_\mathbf{B}$, where $\hat{\tilde{x}}_{Bi}=\left(\tilde{x}_{Bi}-\dfrac{\tilde{x}_{Br}}{y_{rj}}y_{ij}\right)$,

$i \neq r$ and $\hat{\tilde{x}}_{Br} = \frac{\tilde{x}_{Br}}{y_{rj}}$ are the basic variables

**Theorem 3.4.** If $\tilde{\mathbf{x}}_B = B^{-1}\tilde{\mathbf{b}}$ is a basic feasible solution to problem (3.3) with $\tilde{z}_0 \approx \tilde{\mathbf{c}}_B\tilde{\mathbf{x}}_B$ as the value of the objective function and if $\hat{\tilde{\mathbf{x}}}_B$ is another basic feasible solution with $\hat{\tilde{z}} \approx \hat{\tilde{\mathbf{c}}}_B\hat{\tilde{\mathbf{x}}}_B$ obtained by admitting a non-basic column vector $\mathbf{a}_j$ in the basis for which $(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$ and $y_{ij} > 0$ for some $i$, $i \in \{1,2,3,..., m\}$, then $\hat{\tilde{z}} \succeq \tilde{z}_0$.

### 3.2 Unbounded solution

We have seen that for a column vector $\mathbf{a}_j$ of A which is not in B, for which $(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$ and $y_{ij} > 0$, for some i, is alone considered for inserting into the basis. Let us now discuss the situation when there exists an $\mathbf{a}_j$ such that $(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$ and $y_{ij} \le 0$, for all i = 1, 2, 3,…, m. If $\tilde{a} = [a_1, a_2] \succeq \tilde{0}$ and $\lambda > 0$, then $\lambda\tilde{a} = [\lambda a_1, \lambda a_2] \succeq \tilde{0}$. Now $\lambda$ can be made sufficiently large so that $\lambda\tilde{a} \succeq \tilde{b}$ for any interval number $\tilde{b}$. If $\lambda > 0$, $(\tilde{z}_j - \tilde{c}_j) < \tilde{0}$, then $\lambda(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$. Now the proof of the following theorem follows easily.

**Theorem 3.5.** Let $\tilde{\mathbf{x}}_B = B^{-1}\tilde{\mathbf{b}}$ be a basic feasible solution to problem (3.3). If there exist an $\mathbf{a}_j$ of A which is not in B such that $(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$ and $y_{ij} \le 0$, for all i, $i \in \{1,2,3,...,m\}$ then the problem (3.3) has an unbounded solution.

### 3.3 Conditions of Optimality

As in the classical linear programming problems, we can prove that the process of inserting and removing vectors from the basis matrix will lead to any one of the following situations:

(i). there exist j such that $(\tilde{z}_j - \tilde{c}_j) \prec \tilde{0}$, $y_{ij} \le 0$, $i = 1, 2, 3, \cdot\cdot, m$ or

(ii). for all j, $(\tilde{z}_j - \tilde{c}_j) \succeq \tilde{0}$

In the first case, we get an unbounded solution and if the second case occurs, we have an optimal solution.

**Theorem 3.6.** If $\tilde{\mathbf{x}}_B = B^{-1}\tilde{\mathbf{b}}$ is a basic feasible solution to problem (3.3) and if $(\tilde{z}_j - \tilde{c}_j) \succeq \tilde{0}$ for every column $\mathbf{a}_j$ of A, then $\tilde{\mathbf{x}}_B$ is an optimal solution to (3.3).

## 4 Numerical Examples

**Example 4.1**

Let us consider an interval linear programming problem given in [5].

$$\text{Maximize } \tilde{z} = [-20, 50]\tilde{x} + [0, 10]\tilde{y}$$
$$\text{subject to } 10\tilde{x} + 60\tilde{y} \preceq \widetilde{1080}$$
$$10\tilde{x} + 20\tilde{y} \preceq \widetilde{400}$$
$$10\tilde{x} + 10\tilde{y} \preceq \widetilde{240}$$
$$30\tilde{x} + 10\tilde{y} \preceq \widetilde{420}$$
$$40\tilde{x} + 10\tilde{y} \preceq \widetilde{520}$$
$$\text{and } \tilde{x}, \tilde{y} \succeq \tilde{0}$$

We assume the interval numbers as $\widetilde{1080} = [1075, 1085]$, $\widetilde{400} = [395, 405]$, $\widetilde{240} = [238, 242]$, $\widetilde{420} = [417, 423]$ and $\widetilde{520} = [516, 524]$

Now the given ILP becomes

$$\text{Maximize } \tilde{z} = [-20, 50]\tilde{x} + [0, 10]\tilde{y}$$
$$\text{subject to } 10\tilde{x} + 60\tilde{y} \preceq [1075, 1085]$$
$$10\tilde{x} + 20\tilde{y} \preceq [395, 405]$$
$$10\tilde{x} + 10\tilde{y} \preceq [238, 242]$$
$$30\tilde{x} + 10\tilde{y} \preceq [417, 423]$$
$$40\tilde{x} + 10\tilde{y} \preceq [516, 524]$$
$$\text{and } \tilde{x}, \tilde{y} \succeq \tilde{0}$$

Using the results proved in this paper, we solve the ILP as follows:

By introducing non-negative slack variables $\tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5$, the standard form of the given ILP becomes

$$\text{Maximize } \tilde{z} = [-20, 50]\tilde{x} + [0, 10]\tilde{y}$$
$$\text{subject to } 10\tilde{x} + 60\tilde{y} + \tilde{s}_1 \approx [1075, 1085]$$
$$10\tilde{x} + 20\tilde{y} + \tilde{s}_2 \approx [395, 405]$$
$$10\tilde{x} + 10\tilde{y} + \tilde{s}_3 \approx [238, 242]$$
$$30\tilde{x} + 10\tilde{y} + \tilde{s}_4 \approx [417, 423]$$
$$40\tilde{x} + 10\tilde{y} + \tilde{s}_5 \approx [516, 524]$$
$$\text{and } \tilde{x}, \tilde{y}, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3, \tilde{s}_4, \tilde{s}_5 \succeq \tilde{0}.$$

That is
$$\text{Maximize } \tilde{z} = \tilde{\mathbf{c}}\,\tilde{\mathbf{x}}$$
$$\text{subject to } A\tilde{\mathbf{x}} \approx \tilde{\mathbf{b}}$$
$$\text{and } \tilde{\mathbf{x}} \succeq \tilde{\mathbf{0}}$$

where $\tilde{\mathbf{c}} = ([-20, 50], [0, 10], \tilde{0}, \tilde{0}, \tilde{0}, \tilde{0}, \tilde{0})$,

$$A=\begin{bmatrix}10&60&1&0&0&0&0\\10&20&0&1&0&0&0\\10&10&0&0&1&0&0\\30&10&0&0&0&1&0\\40&10&0&0&0&0&1\end{bmatrix},\ \tilde{\mathbf{b}}=\begin{bmatrix}[1075,1085]\\ [395,405]\\ [238,242]\\ [417,423]\\ [516,524]\end{bmatrix}\text{ and }\tilde{\mathbf{x}}=\begin{bmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{s}_1\\ \tilde{s}_2\\ \tilde{s}_3\\ \tilde{s}_4\\ \tilde{s}_5\end{bmatrix}$$

**Initial iteration:**

| | | $\tilde{c}_j$ | $[-20,50]$ | $[0,10]$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $\tilde{X}_B$ | $\tilde{x}$ | $\tilde{y}$ | $\tilde{s}_1$ | $\tilde{s}_2$ | $\tilde{s}_3$ | $\tilde{s}_4$ | $\tilde{s}_5$ | $\theta$ |
| $\tilde{0}$ | $\tilde{s}_1$ | [1075, 1085] | 10 | 60 | 1 | 0 | 0 | 0 | 0 | [107.5,108.5] |
| $\tilde{0}$ | $\tilde{s}_2$ | [395, 405] | 10 | 20 | 0 | 1 | 0 | 0 | 0 | [39.5,40.5] |
| $\tilde{0}$ | $\tilde{s}_3$ | [238,242] | 10 | 10 | 0 | 0 | 1 | 0 | 0 | [23.8,24.2] |
| $\tilde{0}$ | $\tilde{s}_4$ | [417,423] | 30 | 10 | 0 | 0 | 0 | 1 | 0 | [13.9,14.1] |
| $\tilde{0}$ | $\tilde{s}_5$ | [516,524] | **40** | 10 | 0 | 0 | 0 | 0 | 1 | **[12.9,13.1]** |
| | $\tilde{z}_j-\tilde{c}_j$ | | **[-50,20]** | [-10,0] | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | |

Initial basic feasible solution is given by

$\tilde{s}_1=[1075,1085], \tilde{s}_2=[395,405], \tilde{s}_3=[238,242], \tilde{s}_4=[417,423]$ and $\tilde{s}_5=[516,524]$.

**First iteration:**

| | | $c_j$ | $[-20,50]$ | $[0,10]$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $\tilde{X}_B$ | $\tilde{x}$ | $\tilde{y}$ | $\tilde{s}_1$ | $\tilde{s}_2$ | $\tilde{s}_3$ | $\tilde{s}_4$ | $\tilde{s}_5$ | $\theta$ |
| $\tilde{0}$ | $\tilde{s}_1$ | [944,956] | 0 | 115/ 2 | 1 | 0 | 0 | 0 | -1/ 4 | [16.4,16.6] |
| $\tilde{0}$ | $\tilde{s}_2$ | [264,276] | 0 | 35/ 2 | 0 | 1 | 0 | 0 | -1/ 4 | [15.1,15.9] |
| $\tilde{0}$ | $\tilde{s}_3$ | [107,113] | 0 | 15/ 2 | 0 | 0 | 1 | 0 | -1/ 4 | [14.3,24.2] |
| $\tilde{0}$ | $\tilde{s}_4$ | [24,36] | 0 | **5/ 2** | 0 | 0 | 0 | 1 | -3/ 4 | **[9.6,14.4]** |
| [-20,50] | $\tilde{x}$ | [12.9,13.1] | 1 | 1/ 4 | 0 | 0 | 0 | 0 | 1/ 40 | [51.6,52.4] |
| | $(z_j-c_j)$ | | [0,0] | **[-15,12.5]** | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | [-0.5,1.25] | |

The improved basic feasible solution is given by

$\tilde{s}_1=[944,956], \tilde{s}_2=[264,276], \tilde{s}_3=[107,113], \tilde{s}_4=[24,36]$ and $\tilde{x}=[12.9,13.1]$.

**Second iteration:**

| | | $c_j$ | $[-20,50]$ | $[0,10]$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | $Y_B$ | $\tilde{X}_B$ | $\tilde{x}$ | $\tilde{y}$ | $\tilde{s}_1$ | $\tilde{s}_2$ | $\tilde{s}_3$ | $\tilde{s}_4$ | $\tilde{s}_5$ | $\theta$ |
| $\tilde{0}$ | $\tilde{s}_1$ | [116,404] | 0 | 0 | 1 | 0 | 0 | -23 | 17 | |
| $\tilde{0}$ | $\tilde{s}_2$ | [12,108] | 0 | 0 | 0 | 1 | 0 | -2.56 | 5 | |
| $\tilde{0}$ | $\tilde{s}_3$ | [-1,41] | 0 | 0 | 0 | 0 | 1 | -3 | 2 | |
| [0,10] | $\tilde{y}$ | [9.6,14.4] | 0 | 1 | 0 | 0 | 0 | 0.4 | -0.3 | |
| [-20,50] | $\tilde{x}$ | [9.3,10.7] | 1 | 0 | 0 | 0 | 0 | -0.1 | 0.1 | |
| | $(z_j-c_j)$ | | [0,0] | [0,0] | $\tilde{0}$ | $\tilde{0}$ | $\tilde{0}$ | [-5,6] | $[-5,5]\approx\tilde{0}$ | |

Since $(\tilde{z}_j-\tilde{c}_j)\succeq\tilde{0}$ for all j, the current basic feasible solution is optimal. The optimal solution is

$\tilde{s}_1=[944,956], \tilde{s}_2=[264,276], \tilde{s}_3=[107,113], \tilde{s}_4=[24,36]$ and $\tilde{x}=[12.9,13.1]$.

**Example 4.2**

In daily diet, the quantities of certain foods should be taken to meet certain nutritional requirements at a minimum cost. The consideration is limited to bread, butter and milk and to proteins, fats and carbohydrates. The yields of these nutritional requirements per unit of the three types of food are given below:

| | Yield per unit | | |
|---|---|---|---|
| Food type | Proteins | Fats | Carbohydrates |
| Bread | 4 | 1 | 2 |
| Butter | 3 | 2 | 1 |
| Milk | 3 | 2 | 1 |

Also one should have 4- 6 grams of proteins, 1- 3 grams of fats and 2- 4 grams of carbohydrates at least every day. The costs of bread, butter and milk may vary slightly from day to day, but are Rs 1- 3 per gram of bread, Rs 8-10 per gram of butter and Rs 2-4 per gram of milk respectively.

We model this problem as an interval linear programming problem as follows

$$\begin{aligned}&\min\tilde{z}\approx[1,3]\tilde{x}_1+[8,10]\tilde{x}_2+[2,4]\tilde{x}_3\\&\text{subject to }\ 4\tilde{x}_1+3\tilde{x}_2+3\tilde{x}_3\succeq[4,6],\\&\qquad\tilde{x}_1+2\tilde{x}_2+2\tilde{x}_3\succeq[1,3],\\&\qquad 2\tilde{x}_1+\tilde{x}_2+\tilde{x}_3\succeq[2,4]\\&\qquad\text{and }\ \tilde{x}_1\succeq\tilde{0},\tilde{x}_2\succeq\tilde{0},\tilde{x}_3\succeq\tilde{0}.\end{aligned}\tag{3.9}$$

By using the theory developed in this paper we obtain an interval optimal solution,

$\min\tilde{z}\approx\left[-10,\frac{52}{3}\right]=[-10,17.33]$ with $\tilde{x}_1=\left[-1,\frac{11}{3}\right]=[-1,3.67]$

$\tilde{x}_2=[0,0]$ and $\tilde{x}_3=\left[-\frac{4}{3},2\right]=[-1.33,2]$.

## 5 CONCLUSION

We have proved some of the basic theorems related to interval linear programming problems using the generalized interval arithmetic. Applying these results, we have developed a simplex like algorithm for the interval solution of interval linear programming problems without converting them to classical linear programming problems. Numerical examples are also given to illustrate the theory developed in this paper.

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